To derive closed-loop models for coherent feedback control we utilize quantum stochastic differential equations (QSDEs), as in [1,2]. We are working with right-QSDEs and each component in the feedback loop is represented by a triple (S, L, H), where S is a scattering matrix, L is a vector of coupling operators (between input-output fields and internal degrees of freedom) and H is the Hamiltonian of the internal degrees of freedom. We make use of the concatenation product

$$G_2 \boxplus G_1 = \left(\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H_1 + H_2 \right),$$

where the components of G_1 and G_2 need not commute, and the series product

$$G_2 \triangleleft G_1 = (S_2 S_1, S_2 L_1 + L_2, H_1 + H_2 + \mathrm{Im}\{L_2^\dagger S_2 L_1\}).$$

We also note the generalized input-output relations,

$$(d\mathbf{A}_t)_{out} = \mathbf{S} d\mathbf{A}_t + \mathbf{L} dt.$$

We first use the series product to derive an open-loop model for the plant cavity with a coherent driving field. The plant cavity itself is described by an autonomous dynamical model

$$G_b = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_2}\, b \\ \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} \right) = G_{b1} \boxplus G_{b2} \boxplus G_{b3},$$

$$G_{b1} = (1, \sqrt{\kappa_{b1}}\, b, H_{bu}), \quad G_{b2} = (1, \sqrt{\kappa_{b2}}\, b, 0), \quad G_{b3} = (1, \sqrt{\kappa_{b3}}\, b, 0).$$

In order to include a coherent input field β we use the series product,

$$\begin{aligned} N &= G_{b1} \boxplus G_{b2} \boxplus (G_{b3} \triangleleft (1, \beta, 0)) \\ &= G_{b1} \boxplus G_{b2} \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, b^\dagger \beta\}) \\ &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_2}\, b \\ \beta + \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i} (b^\dagger \beta - b \beta^*) \right). \end{aligned}$$

The corresponding open-loop master equation is

$$\dot{\rho} = -i[H,\rho] + \sum_j \left\{ L_j \rho L_j^\dagger - \frac{1}{2} L_j^\dagger L_j \rho - \frac{1}{2} \rho L_j^\dagger L_j \right\}$$

$$\to -i\left[H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*), \rho \right] + (\kappa_{b1} + \kappa_{b2})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2} \rho b^\dagger b \right\}$$

$$+ (\beta + \sqrt{\kappa_{b3}}\, b)\rho(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger) - \frac{1}{2}(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger)(\beta + \sqrt{\kappa_{b3}}\, b)\rho - \frac{1}{2}\rho(\beta^* + \sqrt{\kappa_{b3}}\, b^\dagger)(\beta + \sqrt{\kappa_{b3}}\, b)$$

$$= -i\left[H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*), \rho \right] + (\kappa_{b1} + \kappa_{b2})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2} \rho b^\dagger b \right\}$$

$$+ |\beta|^2 \rho + \beta^* \sqrt{\kappa_{b3}}\, b\rho + \beta \sqrt{\kappa_{b3}}\, \rho b^\dagger + \kappa_{b3} b\rho b^\dagger - \frac{1}{2}|\beta|^2 \rho - \frac{1}{2}\beta^* \sqrt{\kappa_{b3}}\, b\rho - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\, b^\dagger \rho - \frac{1}{2}\kappa_{b3} b^\dagger b \rho$$

$$- \frac{1}{2}|\beta|^2 \rho - \frac{1}{2}\beta^* \sqrt{\kappa_{b3}}\, \rho b - \frac{1}{2}\beta \sqrt{\kappa_{b3}}\, \rho b^\dagger - \frac{1}{2}\kappa_{b3} \rho b^\dagger b$$

$$= -i\left[H_{bu} + \frac{\sqrt{\kappa_{b3}}}{2i}(b^\dagger \beta - b\beta^*), \rho \right] + (\kappa_{b1} + \kappa_{b2} + \kappa_{b3})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2} \rho b^\dagger b \right\} + \frac{1}{2}\beta^* \sqrt{\kappa_{b3}}\,(b\rho - \rho b) - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\,(b^\dagger \rho - \rho b^\dagger).$$

We note that

$$\frac{1}{2}\beta^* \sqrt{\kappa_{b3}}\,(b\rho - \rho b) - \frac{1}{2}\beta \sqrt{\kappa_{b3}}\,(b^\dagger \rho - \rho b^\dagger) = \frac{\sqrt{\kappa_{b3}}}{2}[(\beta^* b - \beta b^\dagger), \rho] = -i\frac{\sqrt{\kappa_{b3}}}{2i}[(\beta b^\dagger - \beta^* b), \rho],$$

hence we can pull this remaining term into the Hamiltonian and finally write

$$\dot{\rho} == -i[H_{bu} - i\sqrt{\kappa_{b3}}\,(b^\dagger \beta - b\beta^*), \rho] + (\kappa_{b1} + \kappa_{b2} + \kappa_{b3})\left\{ b\rho b^\dagger - \frac{1}{2} b^\dagger b \rho - \frac{1}{2}\rho b^\dagger b \right\}.$$

We thus see clearly that the total cavity decay rate is simply $\kappa_b \equiv \kappa_{b1} + \kappa_{b2} + \kappa_{b3}$ while the effects of the driving term can be absorbed into the system Hamiltonian. The driven cavity model can thus be written

$$N_d = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_2}\, b \\ \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} - i\sqrt{\kappa_{b3}}\,(b^\dagger \beta - b\beta^*) \right).$$

The total Hamiltonian here corresponds to H_b in the main text.

We next consider the effects of linear static coherent feedback, with a simple phase shift, as depicted in the upper left panel of Figure 2. We can write

$$
\begin{aligned}
N_{LS} &= G_{b1} \lhd ((e^{i\varphi},0,0) \lhd G_{b2}) \boxplus (G_{b3} \lhd (1,\beta,0)) \\
&= (1, \sqrt{\kappa_{b1}}, H_{bu}) \lhd (e^{i\varphi}, e^{i\varphi}\sqrt{\kappa_{b2}}\, b, 0) \boxplus (G_{b3} \lhd (1,\beta,0)) \\
&= (e^{i\varphi}, (\sqrt{\kappa_{b1}} + e^{i\varphi}\sqrt{\kappa_{b2}})b, H_{bu} + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b) \boxplus (G_{b3} \lhd (1,\beta,0)) \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} (\sqrt{\kappa_{b1}} + e^{i\varphi}\sqrt{\kappa_{b2}})b \\ \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*) + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b \right),
\end{aligned}
$$

where we re-use what we have derived above regarding the driving term, yielding the closed-loop master equation

$$
\dot{\rho} = -i[H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*) + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b] + \left(\kappa_{b3} + |\sqrt{\kappa_{b1}} + e^{i\varphi}\sqrt{\kappa_{b2}}|^2\right)\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\}.
$$

Hence the total cavity decay rate is a function of φ, and there is an additional frequency-pulling term in the Hamiltonian. We note that for $\varphi = 0$ we obtain

$$
\dot{\rho} \to -i[H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*)] + (\kappa_b + 2\sqrt{\kappa_{b1}\kappa_{b2}})\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\},
$$

while for $\varphi = \pi$ we obtain

$$
\dot{\rho} \to -i[H_{bu} - i\sqrt{\kappa_{b3}}(b^\dagger\beta - b\beta^*)] + (\kappa_b - 2\sqrt{\kappa_{b1}\kappa_{b2}})\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\}.
$$

Hence in these simple cases we have either a pure increase or a pure decrease in the cavity decay rate as the only net effect of the feedback. These can be understood as interferometric constructive/destructive interference of the output fields from the κ_{b1} and κ_{b2} cavity mirrors. We infer that since the external driving term (through mirror κ_{b3}) is unaffected, it should be possible to use φ to tune the average intracavity photon number. In particular if we have a detuned driving field, we should be able to decrease the effective driving strength by decreasing the effective κ_b and vice versa.

For the nonlinear dynamic controller we assume two cavities a (controller) and b (plant) with component models

$$G_a = (1, \sqrt{\kappa_a}\, a, H_a),$$

$$G_b = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{\kappa_{b1}}\, b \\ \sqrt{\kappa_{b2}}\, b \\ \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_{bu} \right),$$

where

$$H_a = (\chi_a a^* a + \Delta_a) a^* a,$$

$$H_{bu} = (\chi_b b^* b + \Delta_b) b^* b.$$

We define the partitioning

$$G_b = G_{b1} \boxplus G_{b2} \boxplus G_{b3},$$

where

$$G_{b1} = (1, \sqrt{\kappa_{b1}}\, b, H_{bu}), \quad G_{b2} = (1, \sqrt{\kappa_{b2}}\, b, 0), \quad G_{b3} = (1, \sqrt{\kappa_{b3}}\, b, 0).$$

For the interconnection diagram shown in the upper right panel of Figure 2 we compute the feedback network as

$$\begin{aligned}
N_{ND} &= G_{b1} \triangleleft (G_a \triangleleft ((e^{i\varphi}, 0, 0) \triangleleft G_{b2})) \boxplus (G_{b3} \triangleleft (1, \beta, 0)) \\
&= ((1, \sqrt{\kappa_{b1}}\, b, H_{bu}) \triangleleft ((1, \sqrt{\kappa_a}\, a, H_a) \triangleleft (e^{i\varphi}, e^{i\varphi} \sqrt{\kappa_{b2}}\, b, 0))) \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, \beta b^\dagger\}) \\
&= ((1, \sqrt{\kappa_{b1}}\, b, H_{bu}) \triangleleft (e^{i\varphi}, \sqrt{\kappa_a}\, a + e^{i\varphi} \sqrt{\kappa_{b2}}\, b, H_a + \mathrm{Im}\{e^{i\varphi} \sqrt{\kappa_a \kappa_{b2}}\, a^\dagger b\})) \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, \beta b^\dagger\}) \\
&= \Big(e^{i\varphi}, \sqrt{\kappa_a}\, a + (e^{i\varphi} \sqrt{\kappa_{b2}} + \sqrt{\kappa_{b1}}) b,\ H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b + \mathrm{Im}\{e^{i\varphi} \sqrt{\kappa_a \kappa_{b2}}\, a^\dagger b + \sqrt{\kappa_a \kappa_{b1}}\, a b^\dagger\}\Big) \\
&\quad \boxplus (1, \beta + \sqrt{\kappa_{b3}}\, b, \mathrm{Im}\{\sqrt{\kappa_{b3}}\, \beta b^\dagger\}) \\
&= \left(S_{ND}, \begin{bmatrix} \sqrt{\kappa_a}\, a + (e^{i\varphi} \sqrt{\kappa_{b2}} + \sqrt{\kappa_{b1}}) b \\ \beta + \sqrt{\kappa_{b3}}\, b \end{bmatrix}, H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^\dagger b + \mathrm{Im}\{e^{i\varphi} \sqrt{\kappa_a \kappa_{b2}}\, a^\dagger b + \sqrt{\kappa_a \kappa_{b1}}\, a b^\dagger + \sqrt{\kappa_{b3}}\, \beta b^\dagger\} \right).
\end{aligned}$$

We thus have a total Hamiltonian,

$$H = H_a + H_{bu} + \sin\varphi \sqrt{\kappa_{b1}\kappa_{b2}}\, b^* b + \frac{\sqrt{\kappa_a \kappa_{b2}}}{2i} (e^{i\varphi} a^\dagger b - e^{-i\varphi} a b^\dagger) + \frac{\sqrt{\kappa_a \kappa_{b1}}}{2i} (a b^\dagger - a^\dagger b) + \frac{\sqrt{\kappa_{b3}}}{2i} (\beta b^\dagger - \beta^* b),$$

and (as we did above) we note that the second Lindblad term leads to terms in the Master Equation,

$$
\begin{aligned}
[\dot{\rho}]_{L_2} &= L_2\rho L_2^\dagger - \frac{1}{2}L_2^\dagger L_2\rho - \frac{1}{2}\rho L_2^\dagger L_2 \\
&= (\beta + \sqrt{\kappa_{b3}}\,b)\rho(\beta^* + \sqrt{\kappa_{b3}}\,b^\dagger) - \frac{1}{2}(\beta^* + \sqrt{\kappa_{b3}}\,b^\dagger)(\beta + \sqrt{\kappa_{b3}}\,b)\rho - \frac{1}{2}\rho(\beta^* + \sqrt{\kappa_{b3}}\,b^\dagger)(\beta + \sqrt{\kappa_{b3}}\,b) \\
&= |\beta|^2\rho + \beta\sqrt{\kappa_{b3}}\,\rho b^\dagger + \beta^*\sqrt{\kappa_{b3}}\,b\rho + \kappa_{b3}b\rho b^\dagger - \frac{1}{2}\left(|\beta|^2 + \beta^*\sqrt{\kappa_{b3}}\,b + \beta\sqrt{\kappa_{b3}}\,b^\dagger + \kappa_{b3}b^\dagger b\right)\rho \\
&\quad - \frac{1}{2}\rho\left(|\beta|^2 + \beta^*\sqrt{\kappa_{b3}}\,b + \beta\sqrt{\kappa_{b3}}\,b^\dagger + \kappa_{b3}b^\dagger b\right) \\
&= \kappa_{b3}\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\} + \frac{1}{2}\beta\sqrt{\kappa_{b3}}\,\rho b^\dagger + \frac{1}{2}\beta^*\sqrt{\kappa_{b3}}\,b\rho - \frac{1}{2}\beta\sqrt{\kappa_{b3}}\,b^\dagger\rho - \frac{1}{2}\rho\beta^*\sqrt{\kappa_{b3}}\,b \\
&= \kappa_{b3}\left\{b\rho b^\dagger - \frac{1}{2}b^\dagger b\rho - \frac{1}{2}\rho b^\dagger b\right\} + \frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger)\rho + \frac{\sqrt{\kappa_{b3}}}{2}\rho(\beta b^\dagger - \beta^* b).
\end{aligned}
$$

We retain the first term in braces as a modified $L_2 \to \sqrt{\kappa_{b3}}\,b$ and note that

$$
\begin{aligned}
\frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger)\rho + \frac{\sqrt{\kappa_{b3}}}{2}\rho(\beta b^\dagger - \beta^* b) &= \left[\frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger), \rho\right] \\
&= -i\left[i\frac{\sqrt{\kappa_{b3}}}{2}(\beta^* b - \beta b^\dagger), \rho\right] \\
&= -i\left[\frac{\sqrt{\kappa_{b3}}}{2i}(\beta b^\dagger - \beta^* b), \rho\right].
\end{aligned}
$$

We therefore add this to the original Hamiltonian terms to obtain

$$
\begin{aligned}
H &\to H_a + H_{bu} + \sin\varphi\sqrt{\kappa_{b1}\kappa_{b2}}\,b^* b + \frac{\sqrt{\kappa_a\kappa_{b2}}}{2i}(e^{i\varphi}a^\dagger b - e^{-i\varphi}ab^\dagger) + \frac{\sqrt{\kappa_a\kappa_{b1}}}{2i}(ab^\dagger - a^\dagger b) + i\sqrt{\kappa_{b3}}\,(\beta^* b - \beta b^\dagger), \\
L_1 &= \sqrt{\kappa_a}\,a + (e^{i\varphi}\sqrt{\kappa_{b2}} + \sqrt{\kappa_{b1}}\,)b, \\
L_2 &= \sqrt{\kappa_{b3}}\,b.
\end{aligned}
$$

References

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